PERGAMON

International Journal of Solids and Structures 37 (2000) 5855–5872

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

An analytical solution with local elastoplastic models for the evolution of dynamic softening

X. Xin, Z. Chen*

Department of Civil and Environmental Engineering, University of Missouri, Columbia, MO 65211, USA

Received 8 December 1998; in revised form 20 May 1999

Abstract

Strain softening is associated with the evolution of localization. With the use of the similarity method, a closed-form solution for wave propagation in a strain softening bar is derived in the paper through a partitioned-modeling procedure with local elastoplastic constitutive models. The initial point of the localization is taken as the point at which the type of governing differential equation transforms from a hyperbolic one to an elliptic one due to material softening. The evolution of localization is then represented by a moving material surface between the softening domain and non-softening domain. The motion of the material surface is of diffusion type, representing macroscopically the progressive percolation of heterogeneous flow or microdamage. The evolution of relevant field variables along the bar is shown, and the effects of the model parameters on the solution are discussed to demonstrate the proposed procedure. The analytical solution is unique and stable for the given set of boundary and initial data, and material properties, based on the theory of differential equations. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Local elastoplasticity; Evolution of localization; Diffusion; Similarity method

1. Introduction

Localization is manifested by softening of material properties that is accompanied by large deformations localized in a finite zone. The gradual decline of stress with increasing strain inside the localization zone represents the process of progressive failure or damage, which might result in formation and propagation of macro-cracking through solids.

* Corresponding author. Fax: +1-573-882-4784.
E-mail address: chenzh@missouri.edu (Z. Chen).

PII: S0020-7683(99)00243-7

In the last two decades, considerable efforts have been made to find a suitable approach to model the evolution of inhomogeneous interactions among material particles within the localization zone. Among the continuum approaches proposed are nonlocal models, rate-dependent models and Cosserat continuum models. The nonlocal features have appeared in several forms, which include the strain gradient, imbricated continua, prescribed zones of localization and weighted integral averages of damage. In the nineties, much research on localization has been focused on resolving more efficiently different orientations and patterns of localization zones so that routine applications of localization analyses might be feasible (de Borst et al., 1993; Chen and Schreyer, 1994; Neilsen and Schreyer, 1993; Pietruszczak and Niu, 1993; Pijaudier-Cabot and Benallal, 1993; Vardoulakis et al., 1992; Zbib and Jubran, 1992; among others). As can be seen from the literature review, these models include higher order terms in space and/or time or a localization limiter to predict the evolution of localization. Mathematically speaking, the use of higher order terms makes the governing equations well-posed in a high order sense for localization problems, for which local constitutive models yield ill-posed governing equations. Although many promising results have been obtained by these proposed approaches, there are still some limitations that prohibit the prediction of localization phenomena in a general case. In particular, it is not feasible to perform large-scale computer simulation of structural failure with using higher order models via current computational facilities. Hence, an analytical effort is made in this paper to explore whether the evolution of dynamic softening can be predicted via local constitutive models.

The closed-form solutions for wave propagation in a one-dimensional bar which undergoes strain softening have been discussed by Bazant and Belytschko (1985), Chen and Sulsky (1995) and Armero (1997). The analysis by Bazant and Belytschko (1985) for a classical (local) continuum shows that the length of the strain-softening region tends to localize into a single cross-sectional plane of the bar, at which the strains becomes infinite within an instant. In result, the strain-softening process dissipates no energy in a finite duration if a local model is used, which is not representative of the experimentally observed behavior. By including a localized dissipative mechanism in a local continuum through the strong discontinuities in the displacement field, Armero (1997) obtained an exact closed-form solution with a general localized softening law. However, the influence of softening response on the elastic wave domain is not considered. Chen and Sulsky (1995) suggested that a partitioned-modeling method together with a set of moving jump conditions be applied to the analysis of localization problems without invoking higher order models, based on the previous research (Chen, 1993).

The basic idea of the partitioned-modeling approach is to apply different local constitutive models inside and outside the localization zone, with a moving material surface of discontinuity, which is associated with the local changes of material properties, being defined between different material domains. Consequently, simplified governing differential equations might be formulated for the large-scale computer simulation of structural failure. The partitioned-modeling approach has been used in several cases to obtain approximate solutions for rate-independent elastoplasticity, creep and failure wave problems (Chen, 1993; Chen et al., 1997; Chen and Xin, 1999). An analytical approach, that is different from the previous work via the jump conditions (Chen and Sulsky, 1995), is proposed in this paper to simulate the progressive percolation of heterogeneous flow or microdamage in dynamic failure response. A similarity method is used here to solve the localization problem that involves two moving boundaries within a single domain, without invoking any jump or discontinuity conditions in advance. The major assumptions that the parallelogram rule could be applied to those points on the boundary between hyperbolic and elliptic (softening) domains, and that higher order terms could be omitted to reach a closed-form solutions, which were made in the previous work to accommodate the jump conditions (Chen and Sulsky, 1995), are not required for the exact solutions in this paper. Hence, the proposed analytical approach is rigorous in the mathematical sense. Since the initial point of localization is taken as the point at which the type of governing differential equation transforms from a hyperbolic (elastic) to an elliptic (softening) one, an effort has been made to clarify the definition of boundary and initial conditions associated with different governing equations, and the experimental means to determine model parameters associated with the evolution of localization. The analytical approach proposed here considers not only the response of localized softening zone itself, but also the influence of the softening evolution on the adjacent zone of elastic wave propagation. The key assumption made in this paper is that the motion of the material boundary between hyperbolic and elliptic domains is diffusive. To obtain a closed-form solution, the diffusion speed is assumed to be constant.

The recent results about the delayed failure wave in the glass specimens, that are shocked to near but below the Hugoniot elastic limit (HEL), suggest that the HEL may not be an elastic limit, but rather, may be a transition in failure mechanisms. A possible transition is the one from a delayed kinetic-controlled failure process below the HEL to a prompt stress-controlled failure process above the HEL (Grady, 1995a, 1995b). Based on a careful study on the failure wave phenomenon observed in shocked glasses, it appears that, in the dynamic failure process, microfissuring at one location induces local deformation heterogeneities that in turn initiate microfissuring in the adjacent material and so on, after a critical state is reached (Feng and Chen, 1999). Hence, a diffusion equation governing the progressive percolation of heterogeneous microdamage appears to capture the essence of the dynamic failure evolution in shocked glasses, as verified with the experimental data available. The use of jump conditions could also result in a diffusion equation governing the failure wave speed, through a mathematical argument based on the transition of differential equations (Chen and Xin, 1999). However, a closed-form solution can not be obtained for the nonlinear diffusion equation governing the evolution of microdamage, that depends on the stress state and internal state variables. In order to obtain a closed-form solution in this paper, we assume that the speed of the moving material boundary is constant. This can be thought as a special case of diffusion, i.e., the time average of a real diffusion process. As a result, the model parameters can be explored qualitatively in a closed-form analytical framework. The closed-form solutions obtained by Armero (1997) with the use of a strong discontinuity, also shows that the softening profile propagates at a constant speed along the bar. Due to the limitation of current experimental facilities, it is still a challenging task to quantitatively determine how the internal energy diffuses in real-time associated with the evolution of material failure. However, what is proposed in this paper is not only a possible mathematical approach for modeling material failure with the use of local constitutive models, but also provide a qualitative means to explore the energy dissipation and diffusion with the evolution of material failure from a macroscopic viewpoint, and a useful tool to verify the numerical solutions for nonlinear governing equations.

Based on the theory of differential equations, the analytical solution proposed is unique and stable for given set of boundary and initial data, and material properties. What we want to emphasize here is that a unique and stable solution could be obtained for a given set of data in a controlled manner, just as certain constraint conditions must be imposed on the post-peak solution path to obtain a numerical solution of structural softening responses with other models. In fact, the randomness of the boundary and initial data, and the material properties and defects at different scales play a very important role, and its effects on the softening response should be fully understood before a general approach could be developed without any artificial assumptions. Because of the difficulty involved, we are trying to solve only one problem at a time.

To demonstrate the features of the analytical solution, the evolution of the important field variables along the bar are given together with the effects of the model parameters on the solution. The length of the elastic domain, which is under the influence of the strain-softening response, is found to be affected by the speed of the moving material surface. However, the strain in this elastic domain is characterized by the limit strain ε_L.

2. Analytical solutions

2.1. Approach

The proposed analytical approach is presented first to solve a one-dimensional dynamic softening problem with the use of local elastoplastic constitutive models. In this problem, a tensile bar of length L with mass density ρ is fixed at the left end $x = 0$, as depicted in Fig. 1(a). It is assumed that the material behavior under dynamic loading can be described through a rate-independent linear elastoplastic softening model, in which E denotes Young's modulus, σ_L limit stress with $\varepsilon_L = \sigma_L / E$, and β softening parameter as shown in Fig. 1(b). If σ_a, constant stress, applied at the right end $x = L$, an elastic stress wave will propagate along the bar from $x = L$ to $x = 0$. In order to initiate softening at the fixed end when the stress is doubled, the value of applied stress σ_a is chosen within the range $(\frac{1}{2}\sigma_L, \sigma_L)$. Generally, the equation of motion for one-dimensional wave propagation can be written as

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{\rho}\frac{d\sigma}{d\varepsilon}\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

where ε is normal strain, σ normal stress, t time, and u displacement.

The tangent modulus $d\sigma/d\varepsilon$ of the stress–strain relation is not constant in general. Before the wave front reaches the rigid boundary at $x = 0$, the stress is below the limit stress, so the material behavior is elastic, with the tangent modulus $d\sigma/d\varepsilon = E$. Therefore, the differential equation governing the elastic wave domain is hyperbolic: $u_{tt} - v_e^2 u_{xx} = 0$, with $v_e = \sqrt{E/\rho}$ being the uniaxial elastic wave speed. When the wave front reaches the rigid boundary at $t = t_L = L/v_e$, stress will be doubled and exceed the limit stress. As a result, the material will undergo strain softening, with a negative tangent modulus $d\sigma/d\varepsilon = -\beta E$. Thus, a new domain, i.e., a dynamic strain-softening domain, is produced, in which the type of governing differential equation transforms from a hyperbolic one to an elliptic one: $u_{tt} + \beta v_e^2 u_{xx} = 0$. If nothing is added to regularize the solution, a zero measure of the elliptic domain would occur for the local model (Bazant and Belytschko, 1985). However, the boundary between the elliptic and hyperbolic domain is assumed here to be governed by a diffusion equation which is the transition type between a hyperbolic one and an elliptic one of PDE (John, 1982). As can be seen, the initiation of localization is accompanied by the initiation of a material boundary across which the type of governing differential equation changes due to the material softening. This material boundary will move along the bar during the evolution process of localization. The evolution of an elliptic (softening) domain from a critical point is similar to the evolution of a turbulence (elliptic) domain from a perturbation (Chen and Clark, 1995). The physics behind the evolution of a softening domain is related here to the progressive percolation of heterogeneous flow or microdamage, starting from a critical state.

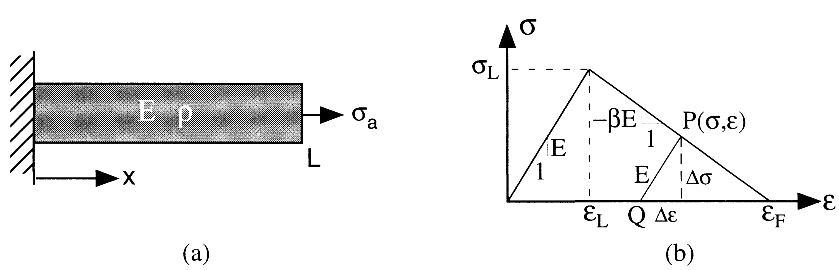

Fig. 1. (a) The configuration of a strain-softening bar under dynamic loading. (b) Local constitutive model: β softening parameter, σ_L limit stress. PQ: unloading path of a material point.

Two facts based on experimental observations should be elucidated here. First, the size of the localization zone is finite. Second, this finite localization zone is not formed within an instant, and instead, it is formed during a finite time span. In other words, the evolution of the localization zone, which is represented by a moving material boundary between softening and non-softening domain, has a finite speed. In reality, the motion of the material boundary depends on the stress state and internal state variables. To obtain a closed-form solution, however, this speed is assumed here to be a constant v_b, which can be thought as a special case of diffusion: the time average of a real diffusion process.

In summary, the whole solution domain is partitioned by a moving boundary $\partial\Omega_I$: $x_b(t) = v_b(t - t_L)$ after the limit state is reached. At any given time $t > t_L$, the whole domain consists of two sub-domains: an elliptic domain Ω_I and a hyperbolic domain $(\Omega_{II} + \Omega_{III})$, as shown in Fig. 2. Therefore, we can apply different local constitutive models to elliptic and hyperbolic domains, respectively, and obtain an analytical solution for the whole domain. For convenience, the governing differential equations are expressed in a strain-based form in the following derivations.

2.2. Solution in the elliptic domain (Ω_I)

Due to the sign change in the tangent modulus, the differential equation governing the localization zone is elliptic:

$$\frac{\partial^2 \varepsilon^-}{\partial t^2} + \frac{\beta E}{\rho}\frac{\partial^2 \varepsilon^-}{\partial x^2} = 0 \quad x \in \left[0, x_b(t)\right] \text{ and } t \in [t_L, t_F] \tag{2}$$

where the superscript $^-$ denotes the field variables to the left of the moving material boundary, and the location of moving material boundary $x_b(t)$ is defined as

$$x_b(t) = v_b(t - t_L) \tag{3}$$

From a mathematical viewpoint, Eq. (2) can be further thought as a Laplace equation, because the

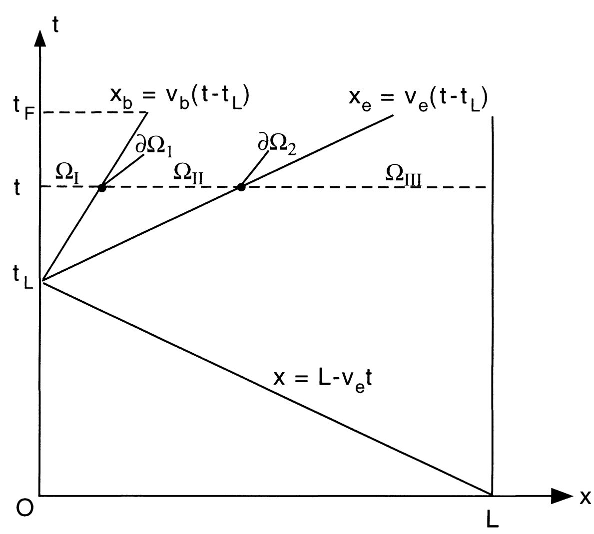

Fig. 2. Solution domain partitioned by a moving boundary $\delta\Omega_I$: $x_b = v_b(t - t_L)$. At any time $t > t_L$, the whole domain is partitioned into two domains: an elliptic domain Ω_I and a hyperbolic domain $(\Omega_{II} + \Omega_{III})$.

constant coefficient $\beta E/\rho$ can be scaled into the x-dimension. It is known that a Laplace equation signifies a potential distribution field. From a physical viewpoint, the evolution of localization is considered here to be a diffusion process, in which the energy carried by the elastic stress wave is dissipated in the localization zone that results in the progressive percolation of heterogeneous flow or microdamage.

The following set of data is prescribed for Eq. (2). First, the strain at the rigid boundary attains the limit strain ε_L at the moment when the elastic wave front arrives at the fixed end:

$$\varepsilon^-(x=0, t=t_L) = \varepsilon_L \tag{4a}$$

Second, the strain just across the moving boundary $x = x_b(t)$ is kept as a constant ε_L at any time, which represents the initiation of softening at that material surface:

$$\varepsilon^-(x=x_b(t), t) = \varepsilon_L \quad t \in [t_L, t_F] \tag{4b}$$

At the final stage when $t = t_F$, the strain at the fixed end attains ε_F at which the bar will lose its load-carrying capacity so that the continuum theory does not hold any more.

$$\varepsilon^-(x=0, t=t_F) = \varepsilon_F \tag{4c}$$

where ε_F is defined as follows according to the linear softening law:

$$\varepsilon_F = \left(1 + \frac{1}{\beta}\right)\varepsilon_L \tag{5}$$

The problem consisting of Eq. (2), boundary and initial condition (4) and given material properties is a Dirichlet problem for the Laplace equation. A solution for Eq. (2) may be expressed as follows:

$$\varepsilon^- = A + B(t - t_L) + Cx + D\arctan\left[\frac{x}{\sqrt{\beta}v_e(t - t_L)}\right] \tag{6}$$

with A, B, C and D being the constants to be determined from the boundary and initial data.

By the initial condition (4a), we have $A = \varepsilon_L$. Substituting this result and Eq. (6) into the moving boundary condition (4b) and rearranging the terms, we obtain the following relation:

$$(B + Cv_b)t + D\arctan\left[\frac{v_b}{\sqrt{\beta}v_e}\right] - (B + Cv_b)t_L = 0 \tag{7a}$$

Since t is arbitrary, we must have

$$B + Cv_b = 0 \quad \text{and} \quad D = 0. \tag{7b}$$

Substituting Eq. (7b) into Eq. (6), we have

$$\varepsilon^- = \varepsilon_L + B(t - t_L) - \frac{B}{v_b}x \tag{7c}$$

By the condition (4c) when failure occurs, we can obtain

$$B = \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} \tag{7d}$$

Therefore, the solution for this problem is given by

$$\varepsilon^{-}(x,t) = \varepsilon_L + \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}\left(t - t_L - \frac{x}{v_b}\right) \quad t \in [t_L, t_F] \text{ and } x \in \left[0, v_b(t - t_L)\right] \tag{8a}$$

This solution has following properties:

1. When $t = t_L$, localization initiates at the point $x = 0$ with $\varepsilon = \varepsilon_L$.
2. At any position x, strain is increasing with time t.
3. At any time t, the strain is decreasing with x.
4. When $t = t_F$, strain at the fixed boundary $x = 0$ attains ε_F.

The corresponding stress field is calculated through the linear softening law as follows:

$$\sigma^{-}(x,t) = \sigma_L - \beta E\left[\varepsilon^{-}(x,t) - \varepsilon_L\right] \tag{8b}$$

From the theory of partial differential equations, two remarks should be made here:

1. This solution is unique for the given Dirichlet problem.
2. As expected by the maximum principle, the maximum and minimum values attain at the boundary (McOwen, 1996).

The corresponding displacement and velocity fields then take the forms of

$$u(x,t) = \int_0^{x \le x_b(t)} \varepsilon(x,t)\,\mathrm{d}x = \varepsilon_L x + \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}\left((t - t_L)x - \frac{x^2}{2v_b}\right) \tag{8c}$$

$$v(x,t) = \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} x \quad x \in \left[0, v_b(t - t_L)\right] \tag{8d}$$

2.3. Solution in the hyperbolic domain ($\Omega_{II} + \Omega_{III}$)

The governing differential equation for the hyperbolic domains is given by

$$\frac{\partial^2 \varepsilon^{+}}{\partial t^2} - \frac{E}{\rho}\frac{\partial^2 \varepsilon^{+}}{\partial x^2} = 0 \quad x \in \left[x_b(t), L\right] \text{ and } t \in [t_L, t_F] \tag{9}$$

As can be seen from Fig. 2, through the moving material boundary $\partial\Omega_1$ the softening response only affects the solution at the points between the moving boundary $\partial\Omega_1 = x_b(t)$ and moving boundary $\partial\Omega_2 = x_e(t)$ that are originated at the point $(0, t_L)$, recalling the concepts of domain of dependence and domain of influence (John, 1982; McOwen, 1996). Therefore, the hyperbolic domain can be further divided into two sub-domains: $\Omega_{II} = \{(x, t): x \in [x_b(t), x_e(t)], t \in [t_L, t_F]\}$ and $\Omega_{III} = \{(x, t): x \in [x_e(t), L], t \in [t_L, t_F]\}$, where the sub-domain Ω_{II} is under the influence of softening response but the sub-domain Ω_{III} is not.

Because of the evolution of localization zone and the propagation of reflected wave, the boundaries both at the left and right side of the sub-domain Ω_{II} is moving with time. The left boundary $\partial\Omega_1$ between softening domain Ω_I and non-softening domain Ω_{II}, is assumed to be moving at a constant speed v_b and characterized by the limit strain ε_L. The right boundary $\partial\Omega_2$ between sub-domain Ω_{II} and Ω_{III} is moving with speed v_e and characterized by a bounded strain ε_u which must be less than ε_L. Thus, for the sub-domain Ω_{II}, we have the following set of prescribed conditions:

$$\varepsilon^{+}(x = x_b(t), t) = \varepsilon_L \quad t \in [t_L, t_F] \tag{10a}$$

$$\varepsilon^{+}(x = x_{\mathrm{e}}(t), t) = \varepsilon_{\mathrm{u}} \quad t \in [t_{\mathrm{L}}, t_{\mathrm{F}}] \tag{10b}$$

Since the given data on the moving material boundary $\partial\Omega_1$ are the time-dependent solutions of an elliptic equation, the parallelogram rule can not be applied here to solve the given wave equation in sub-domain Ω_{II}. To deal with the specific problem of wave propagation in which two moving boundaries exist within a single domain, a similarity method (Bluman and Kumei, 1989; Olver, 1993) is used. We define a similarity variable as follows:

$$\eta = \frac{x - x_{\mathrm{b}}(t)}{x_{\mathrm{e}}(t) - x_{\mathrm{b}}(t)} \quad x \in \left[x_{\mathrm{b}}(t), x_{\mathrm{e}}(t)\right] \tag{11}$$

where $x - x_{\mathrm{b}}(t)$ is the distance from the concerned point x to the moving material boundary $\partial\Omega_1$, and $x_{\mathrm{e}}(t) - x_{\mathrm{b}}(t)$ is the current total length of the domain Ω_{II}.

The positions of the left and right boundaries at any time $t > t_{\mathrm{L}}$ are determined by Eq. (3) and the following equation, respectively

$$x_{\mathrm{e}}(t) = v_{\mathrm{e}}(t - t_{\mathrm{L}}) \tag{12}$$

Thus, the similarity variable can be explicitly expressed as

$$\eta = \frac{x - v_{\mathrm{b}}(t - t_{\mathrm{L}})}{(v_{\mathrm{e}} - v_{\mathrm{b}})(t - t_{\mathrm{L}})} \tag{13}$$

We also introduce a dimensionless strain ratio θ as below:

$$\theta = \frac{\varepsilon}{\varepsilon_{\mathrm{L}}} \tag{14}$$

The governing equation for θ is identical with that for ε:

$$\frac{\partial^2\theta}{\partial t^2} - \frac{E}{\rho}\frac{\partial^2\theta}{\partial x^2} = 0 \tag{15}$$

but the conditions on θ are simpler and given by

$$\theta(x = x_{\mathrm{b}}(t), t) = 1 \quad t \in [t_{\mathrm{L}}, t_{\mathrm{F}}] \tag{16a}$$

$$\theta(x = x_{\mathrm{e}}(t), t) = \frac{\varepsilon_{\mathrm{u}}}{\varepsilon_{\mathrm{L}}} \leq 1 \quad t \in [t_{\mathrm{L}}, t_{\mathrm{F}}] \tag{16b}$$

It is reasonable to suppose that θ is not a function of t and x separately, but rather it is a function of the dimensionless ratio η, based on the theorem in dimensional analysis.

The Eq. (15) and condition (16) can then be rewritten in terms of η. This requires that we represent the partial derivatives of θ with respect to t and x in terms of the derivatives with respect to η, which can be found, using chain rule, to be

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{(v_{\mathrm{e}} - v_{\mathrm{b}})^2(t - t_{\mathrm{L}})^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} \tag{17a}$$

$$\frac{\partial^2\theta}{\partial t^2} = \frac{1}{(v_{\mathrm{e}} - v_{\mathrm{b}})^2(t - t_{\mathrm{L}})^2}\left[\frac{x^2}{(t - t_{\mathrm{L}})^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} + \frac{2x(v_{\mathrm{e}} - v_{\mathrm{b}})}{(t - t_{\mathrm{L}})}\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right] \tag{17b}$$

From Eq. (13), we also know that

$$\frac{x}{(t-t_L)} = (v_e - v_b)\eta + v_b \tag{18}$$

Hence, the second order partial differential equation (15) can be reduced to a second order ordinary differential equation

$$\left[(v_e+v_b) - 2v_b\eta - (v_e-v_b)\eta^2\right]\frac{d^2\theta}{d\eta^2} - 2\left[v_b + (v_e-v_b)\eta\right]\frac{d\theta}{d\eta} = 0 \tag{19}$$

by substituting Eqs. (17) and (18) in Eq. (15).

The corresponding boundary locations are simplified as follows: $x = x_b(t)$ maps into $\eta = 0$ and $x = x_e(t)$ maps into $\eta = 1$. Thus, the condition (16) becomes

$$\theta(\eta = 0) = 1 \tag{20a}$$

$$\theta(\eta = 1) = \frac{\varepsilon_u}{\varepsilon_L} \leq 1 \tag{20b}$$

As a result, the original problem is changed from a moving boundary problem to a fixed boundary problem. The fact that the introduction of the similarity variable reduces the partial differential equation (15) to an ordinary differential equation (19) with respect to η and reduces the separate conditions in t and x to consistent conditions involving η alone, is a posterior proof of the validity of the approach.

Let $\Phi = \frac{d\theta}{d\eta}$, then $\frac{d^2\theta}{d\eta^2} = \frac{d\Phi}{d\eta}$. Eq. (19) can therefore be rewritten as

$$\left[(v_e+v_b) - 2v_b\eta - (v_e-v_b)\eta^2\right]\frac{d\Phi}{d\eta} - 2\left[v_b + (v_e-v_b)\eta\right]\Phi = 0 \tag{21}$$

That is,

$$\frac{d\Phi}{\Phi} = \frac{2v_b + 2(v_e-v_b)\eta}{(v_e+v_b) - 2v_b\eta - (v_e-v_b)\eta^2}d\eta \tag{22}$$

Integration of Eq. (22) results in

$$\ln\Phi = -\ln\left[(v_e+v_b) - 2v_b\eta - (v_e-v_b)\eta^2\right] + \ln C_1 \tag{23}$$

where $\ln C_1$ is a constant of integration. It follows from Eq. (23) that

$$\Phi = \frac{C_1}{(v_e+v_b) - 2v_b\eta - (v_e-v_b)\eta^2} = \frac{d\theta}{d\eta} \tag{24}$$

Upon integrating Eq. (24), we can obtain

$$\theta(\eta) = \frac{C_1}{2v_e}\ln\left[\frac{v_e+v_b+(v_e-v_b)\eta}{v_e-v_b-(v_e-v_b)\eta}\right] + C_2 \tag{25}$$

where C_2 is another constant of integration. According to condition (20), we have

$$\theta(\eta = 0) = \frac{C_1}{2v_e}\ln\left[\frac{v_e + v_b}{v_e - v_b}\right] + C_2 = 1 \tag{26a}$$

and

$$\theta(\eta = 1) = \frac{C_1}{2v_e}\ln[\infty] + C_2 = \frac{\varepsilon_u}{\varepsilon_L} \leq 1 \tag{26b}$$

Since (26b) is bounded, it follows that $C_1 = 0$ and $C_2 = 1$. Then, with the use of Eq. (14), the solution of the original problem takes the form of

$$\varepsilon^+(x, t) = \varepsilon_L \quad t \in [t_L, t_F] \text{ and } x \in \left[v_b(t - t_L), v_e(t - t_L)\right] \tag{27a}$$

The above solution for sub-domain Ω_{II} has the following properties:

1. When $t = t_L$, localization initiates at the point $x = 0$ with $\varepsilon = \varepsilon_L$, and sub-domain Ω_{II} starts expanding.
2. The strain is constant inside the domain Ω_{II}.

The corresponding stress, displacement and velocity fields can be found to be

$$\sigma^+(x, t) = E\varepsilon^+(x, t) = \sigma_L \tag{27b}$$

$$u(x, t) = u(x_b(t), t) + \int_{x_b(t)}^{x \leq x_e(t)} \varepsilon(x, t)\,dx = \varepsilon_L x + \frac{v_b}{2}\frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}(t - t_L)^2 \tag{27c}$$

$$v(x, t) = \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} v_b(t - t_L) \tag{27d}$$

The solution of the sub-domain Ω_{III} can be obtained by a similar procedure, with the given boundary conditions

$$\varepsilon(x = x_e(t), t) = \varepsilon_u \quad t \in [t_L, t_F] \tag{28a}$$

$$\varepsilon(x = L, t) = \varepsilon_a = \frac{\sigma_a}{E} \quad t \in [t_L, t_F] \tag{28b}$$

A simple similarity variable η^* and the dimensionless ratio θ^* can be defined, respectively, as follows:

$$\eta^* = \frac{x - v_e(t - t_L)}{L - v_e(t - t_L)} \tag{29}$$

with $L - x_e(t)$ being the current total length of the domain Ω_{III}, and

$$\theta^* = \frac{\varepsilon}{\varepsilon_a} \tag{30}$$

with ε_a being the strain corresponding to the incident elastic stress wave.

The solution can be found to be

$$\varepsilon^+(x, t) = \frac{\sigma_a}{E} \quad t \in [t_L, t_F] \text{ and } x \in \left[v_e(t - t_L), L\right] \tag{31a}$$

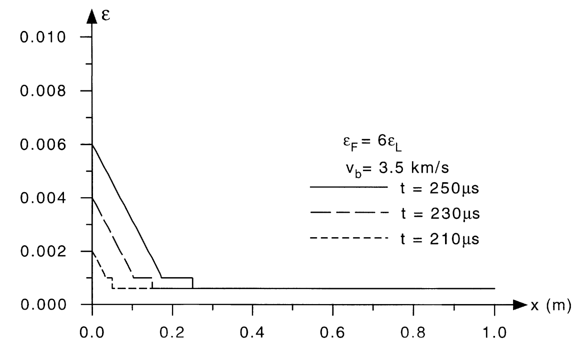

Fig. 3. The evolution of localization after the limit state is reached.

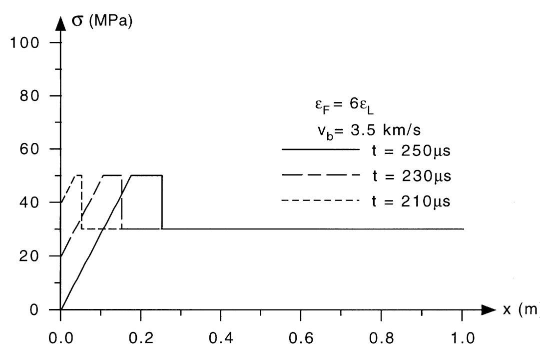

Fig. 4. The decrease of stress corresponding to Fig. 3

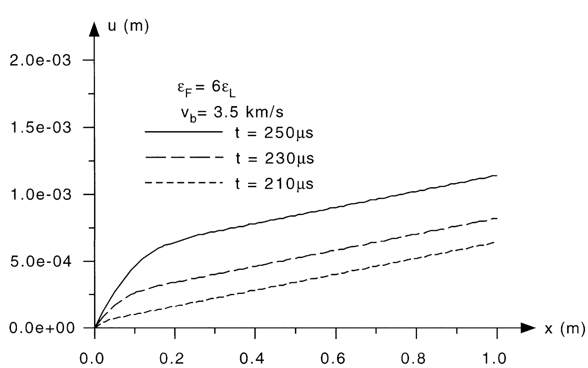

Fig. 5. The changes of displacement corresponding to Fig. 3.

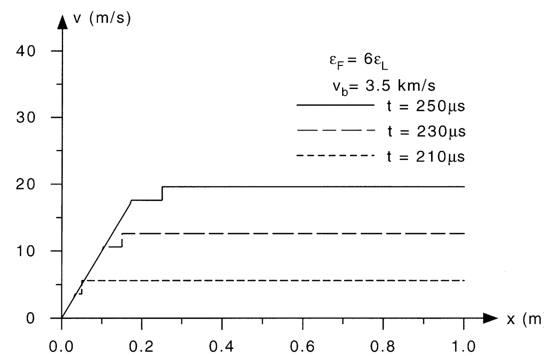

Fig. 6. The changes of velocity corresponding to Fig. 3.

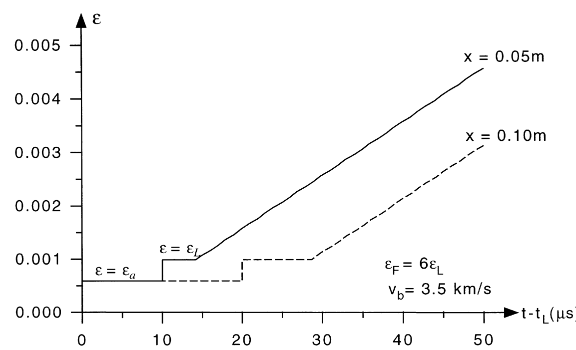

Fig. 7. The strain history at $x = 0.05$ and 0.1m after the localization occurs.

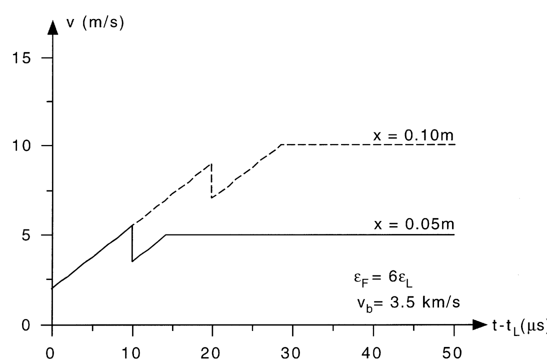

Fig. 8. The velocity history at $x = 0.05$ and 0.1 m corresponding to Fig. 7.

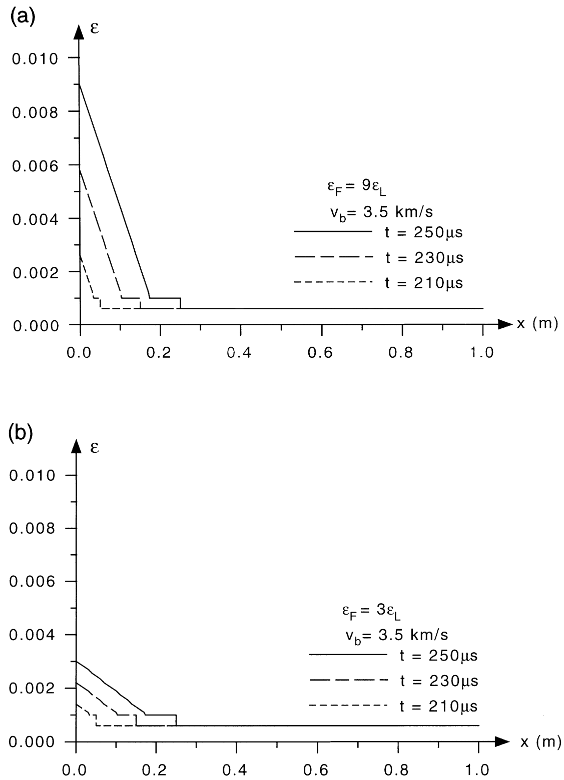

Fig. 9. (a) The evolution of localization after the limit state is reached for a larger ε_F. (b) The evolution of localization after the limit state is reached for a smaller ε_F.

The corresponding stress, displacement and velocity fields take the forms of

$$\sigma^+(x, t) = E\varepsilon^+(x, t) = \sigma_a \tag{31b}$$

$$u(x, t) = u(x_e(t), t) + \int_{x_e(t)}^{x \le L} \varepsilon(x, t)\,dx = \varepsilon_e x + v_e(\varepsilon_L - \varepsilon_e)(t - t_L) + \frac{v_b}{2}\frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}(t - t_L)^2 \tag{31c}$$

$$v(x, t) = v_e(\varepsilon_L - \varepsilon_e) + \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} v_b(t - t_L) \tag{31d}$$

Obviously, there is a jump in strain across the moving boundary $\partial\Omega_2$ between Ω_{II} and Ω_{III}. This strain jump has a magnitude of $(\varepsilon_L - \varepsilon_a)$. Correspondingly, there is also a velocity discontinuity with a magnitude of v_e $(\varepsilon_L - \varepsilon_a)$ across the moving boundary $\partial\Omega_2$.

2.4. Remarks on the solutions

By Eq. (8a), the strain rate in the localization zone is constant and given by

$$\dot{\varepsilon}^{-}(x,t) = \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} \quad t \in [t_L, t_F] \text{ and } x \in [0, x_b(t)] \tag{32}$$

In the elastic domain, the strain rate is zero,

$$\dot{\varepsilon}^{+}(x,t) = 0 \quad t \in [t_L, t_F] \text{ and } x \in [x_b(t), L] \tag{33}$$

Thus, there is a jump in strain rate across the moving material boundary $\partial\Omega_1 = x_b(t)$, as expected in the conventional definition of a localized failure mode (Chen, 1996).

$$\dot{\varepsilon}^{-}(x = x_b(t), t) \neq \dot{\varepsilon}^{+}(x = x_b(t), t) \quad t \in [t_L, t_F] \tag{34}$$

However, the strain is continuous across this boundary:

$$\varepsilon^{-}(x = x_b(t), t) = \varepsilon^{+}(x = x_b(t), t) = \varepsilon_L \quad t \in [t_L, t_F] \tag{35}$$

Consequently, displacement u must also be continuous across this boundary.

Based on the local constitutive model (8b), the permanent strain can be calculated as follows

$$\varepsilon_P(x) = \varepsilon - \Delta\varepsilon = (1 + \beta)(\varepsilon - \varepsilon_L) \tag{36}$$

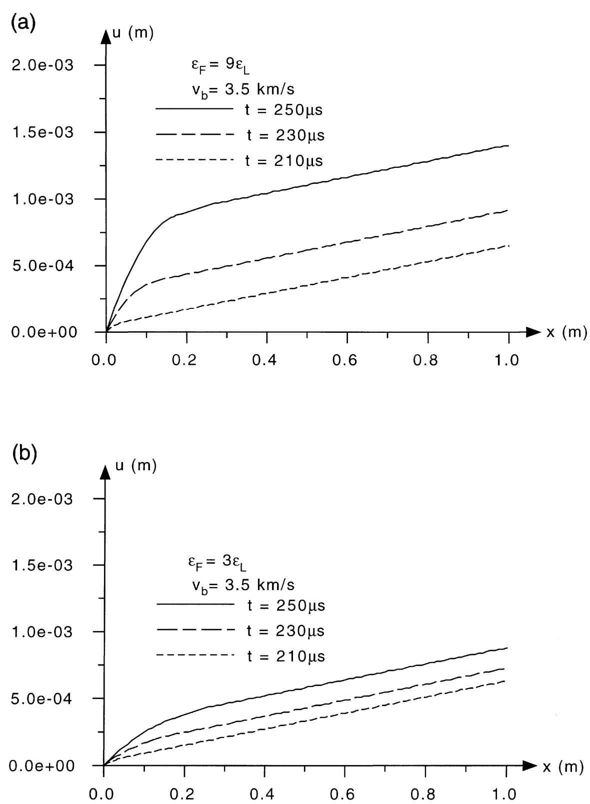

Fig. 10. (a) The displacement distribution corresponding to Fig. 9(a). (b) The displacement distribution corresponding to Fig. 9(b).

Substituting Eq. (8a) in Eq. (36), the permanent strain in the localization zone, after the load-carrying-capacity of the bar is lost at which $\varepsilon(x=0,\ t=t_\mathrm{F})=\varepsilon_\mathrm{F}$ as shown in Fig. 1(b), can be expressed as

$$\varepsilon_\mathrm{P}(x)=\left[1-\frac{x}{v_\mathrm{b}(t_\mathrm{F}-t_\mathrm{L})}\right]\varepsilon_\mathrm{F} \tag{37}$$

with ε_F being defined in Eq. (5).

The permanent elongation can then be obtained by the integration of Eq. (37):

$$\Delta L=\int_0^{v_\mathrm{b}(t_\mathrm{F}-t_\mathrm{L})}\varepsilon_\mathrm{P}(x)\,\mathrm{d}x=\frac{1}{2}\varepsilon_\mathrm{F}v_\mathrm{b}(t_\mathrm{F}-t_\mathrm{L}) \tag{38}$$

Hence, the speed of moving boundary v_b (an average value during the time span $t\in[t_\mathrm{L},t_\mathrm{F}]$) can be estimated as follows, if ε_F, t_F and the permanent elongation ΔL can be measured through experiments,

$$v_\mathrm{b}=\frac{2\Delta L}{\varepsilon_\mathrm{F}(t_\mathrm{F}-t_\mathrm{L})} \tag{39}$$

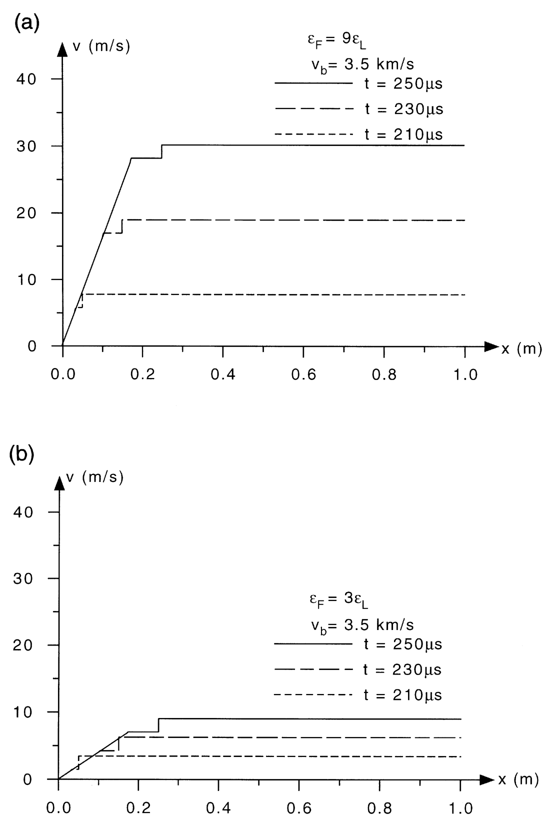

Fig. 11. (a) The velocity distribution corresponding to Fig. 9(a). (b) The velocity distribution corresponding to Fig. 9(b).

2.5. Demonstration

To demonstrate the features of the proposed analytical solution, the material parameters are assigned as follows which are representative of concrete: $E = 50$ GPa, $\rho = 2000$ kg/cm^3, $\sigma_L = 50$ MPa. The other parameters have the value of $L = 1$ m, $\sigma_a = 30$ Pa, $t_F = 250$ μs.

For comparison purpose, the softening parameter $\beta = 0.125$, 0.2, 0.5 are used, correspondingly, $\varepsilon_F = 9\varepsilon_L$, $6\varepsilon_L$, $3\varepsilon_L$ based on Eq. (5). The speed of moving material surface $v_b = \alpha v_e$, with $\alpha = 0.5$, 0.7, 0.9 being used for three different cases.

Since the incident stress wave has a level of $\sigma_a = 30$ MPa, the doubled stress at the fixed end will exceed the limit stress $\sigma_L = 50$ MPa. Hence, the localization will occur at $t = t_L$. As shown in Fig. 3 and Fig. 7 for the case of $\beta = 0.2$, there is a jump in the strain rate across the moving material boundary $\partial\Omega_1$ during the evolution of localization. The corresponding decrease of stress behind the moving material boundary is shown in Fig. 4. The changes in strain distribution are also reflected through the changes in displacement and velocity in Figs. 5 and 6. The velocity jump across the moving boundary $\partial\Omega_2$ is shown in Fig. 8. It can be verified that the jump value is given by $\Delta v = -v_e\Delta\varepsilon$. The effects of the softening parameter β on the strain, displacement and velocity fields are displayed in Figs. 9–11. Figs. 12 and 13 demonstrate the effects of v_b on the strain, displacement and velocity fields. As can be seen,

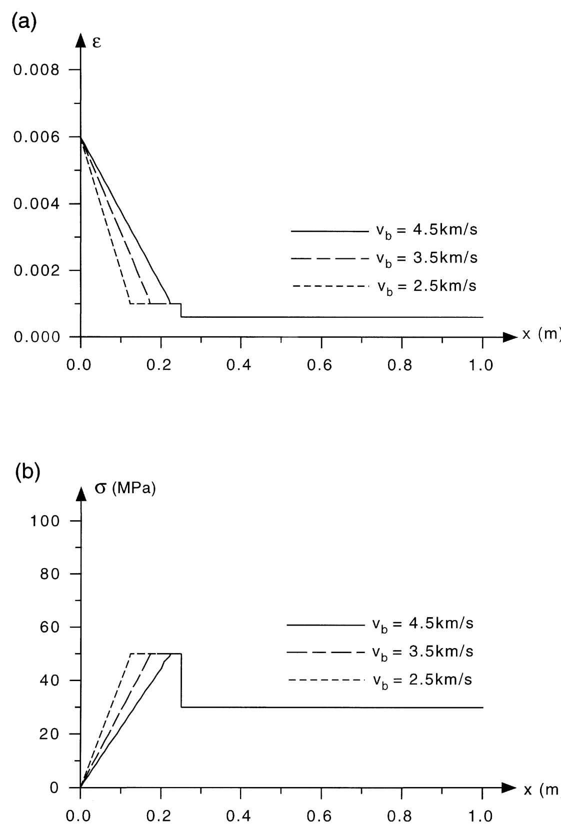

Fig. 12. (a) The strain distribution at $t = 250$ μs with different speeds of the moving boundary for $\varepsilon_F = 6\varepsilon_L$. (b) The stress distribution at $t = 250$ μs corresponding to Fig. 12(a).

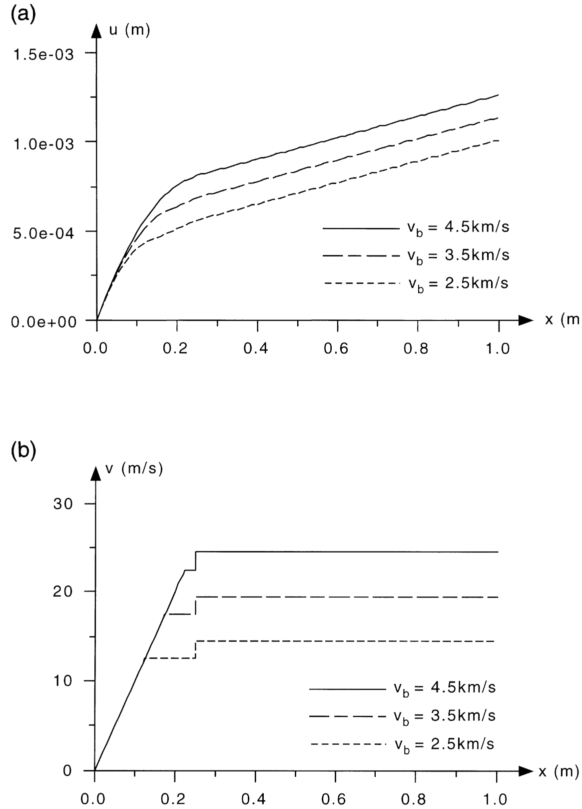

Fig. 13. The displacement distribution at $t = 250$ μs coresponding to Fig. 12(a). (b) The velocity distribution at $t = 250$ μs corresponding to Fig. 12(a).

the analytical solutions represent the essential features of softening with localization, and are continuously dependent on the given data.

3. Conclusions

To predict the evolution of localization due to dynamic softening, something must be added to the local constitutive models to regularize the solution. Instead of invoking higher order term in space and/or time, a rigorous partitioned-modeling approach is employed to obtain a closed-form solution for a dynamic softening bar with local elastoplastic models, via a similarity method. The similarity method is a useful tool to study the problem involving moving material boundary. It may reduce the second order PDE to a second order ODE and change the moving boundary conditions to fixed boundary conditions. The important field variables along the bar are shown, and the effects of model parameters on the solutions are investigated to demonstrate the proposed analytical approach. The analytical solutions for the bar problem under the given set of boundary and initial data, and material properties are unique and stable based on the theory of differential equations, and also verify the jump conditions in the field variables. The analytical approach provides a useful estimate for the diffusion speed of the moving material boundary between softening and non-softening domains.

To obtain a closed-form solution, the key assumption made in this paper is that the material surface is moving at a constant speed. Based on the experimental data available, however, the evolution of localization appears to be an energy dissipation and diffusion process which should be dependent on the stress state and internal state variables. A computational procedure must be developed to simulate the nonlinear diffusion process associated with the evolution of material failure. The closed-form solution obtained here could be used to verify the computational procedure.

Acknowledgements

This work was sponsored in part by the Air Force Office of Scientific Research, USAF with Maj. M. Chipley being program director, and by National Science Foundation with Dr. K. Chong being program director. The authors are grateful to Prof. S. Dostoglou in the Department of Mathematics at the University of Missouri-Columbia for valuable discussion. The authors are also grateful to the reviewers for discerning comments on this paper.

References

Armero, F., 1997. On the characterization of localized solutions in inelastic solids: an analysis of wave propagation in a softening bar. Technical Report UCB/SEMM-97/18, University of California-Berkeley.

Bazant, Z.P., Belytschko, T.B., 1985. Wave propagation in a strain-softening bar: exact solution. J. Engng. Mech 111, 381–389.

Bluman, G.W., Kumei, S., 1989. Symmetries and Differential Equations. Springer–Verlag, New York, NY, USA.

de Borst, R., Sluys, L.J., Mühlhaus, H.B., Pamin, J., 1993. Fundamental issues in finite element analyses of localization of deformation. Engng. Comput 10, 99–121.

Chen, Z., 1993. A partitioned-solution method with moving boundaries for nonlocal plasticity. In: Kolymbas, D. (Ed.), Modern Approaches to Plasticity. Elsevier, New York, NY, USA, pp. 449–468.

Chen, Z., 1996. Continuous and discontinuous failure modes. J. Engng. Mech 122, 80–82.

Chen, Z., Clark, T., 1995. Some remarks on domain-transition problems. Archives of Mechanics 47, 499–512.

Chen, Z., Schreyer, H.L., 1994. On nonlocal damage models for interface problems. Int. J. Solids Structures 31, 1241–1261.

Chen, Z., Sulsky, D., 1995. A partitioned-modeling approach with moving jump conditions for localization. Int. J. Solids Structures 32, 1893–1905.

Chen, Z., Wang, M.L., Lu, T., 1997. Study of tertiary creep of rock salt. J. Engng. Mech 123, 77–82.

Chen, Z., Xin, X., 1999. An analytical and numerical study of failure waves. Int. J. Solids Structures 36, 3977–3991.

Feng, R., Chen, Z., 1999. Propagation of heterogeneous microdamage in shocked glasses. In: Goncalves, P., Jasiuk, I., Pamplona, D., Steele, C., Weber, H., Bevilacqua, L. (Eds.), Applied Mechanics in the Americas, vol. 7. American Academy of Mechanics, pp. 691–694.

Grady, D.E., 1995a. Dynamic properties of ceramic materials. Technical Report SAND94-3266, Sandia National Laboratories, Albuquerque, NM.

Grady, D.E., 1995b. Shock properties of high-strength ceramics. Technical Memorandum TMD G0395, Sandia National Laboratories, Albuquerque, NM.

John, F., 1982. Partial Differential Equations. Springer–Verlag, New York, NY, USA.

McOwen, R.C., 1996. Partial Differential Equations: Methods and Applications. Prentice-Hall, Upper Saddle River, NJ, USA.

Neilsen, M.K., Schreyer, H.L., 1993. Bifurcations in elastic–plastic materials. Int. J. Solids Structures 30, 521–544.

Olver, P.J., 1993. Applications of Lie Group to Differential Equations. Springer–Verlag, New York, NY, USA.

Pietruszczak, S., Niu, X., 1993. On the description of localized deformation. Int. J. Num. Analyt. Meth. Geomech 17, 791–805.

Pijaudier-Cabot, G., Benallal, A., 1993. Strain localization and bifurcation in a nonlocal continuum. Int. J. Solids Structures 30, 1761–1775.

Vardoulakis, I., Shah, K.R., Papanastasiou, P., 1992. Modeling of tool–rock shear interfaces using gradient-dependent flow theory of plasticity. Int. J. Rock Mech. Min. Sci. Geomech. Abstr 29, 573–582.

Zbib, H.M., Jubran, J.S., 1992. Dynamic shear banding: a three dimensional analysis. Int. J. Plast 8, 619–641.